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One-Codimensional Tchebycheff Subspaces

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A necessary and sufficient condition for a finite dimensional Tchybycheff space of real functions on a real set containing its endpoints to contain a 1-codimensional Tchebycheff subspace is given. Examples of n-dimensional Tchebycheff spaces on closed intervals that do not contain $(n-1)$-dimensional Tchebycheff subspaces are given for all $n \geqslant 3$.

1. INTRODUCTION

By a well-known theorem of Krein, every Tchebycheff space (T-space) of real-valued functions defined on an open interval contains a 1-codimensional T-subspace. In [2] we considered T-spaces on real sets containing at most one endpoint, and gave a necessary and sufficient condition for these T-spaces to contain 1-codimensional T-subspaces.

In this paper we discuss this property for T-spaces on real sets containing both endpoints.

DEFINITION 1 ([1]). The set of the real-valued functions $f_1, f_2,..., f_k$, defined on a real set M, is called a Tchebycheff system (T-system) if

$$U\begin{pmatrix} f_1, f_2,..., f_k \\ t_1, t_2,..., t_k \end{pmatrix} = \det(f_i(t_j))_{i,j=1}^{k} \tag{1}$$

has a constant sign for all $t_1, t_2,..., t_k \in M$ with $t_1 < t_2 < \cdots < t_k$.

If L is the span of a T-system, then it is called a T-space. Clearly, every basis of a T-space is a T-system.

An equivalent definition of a T-space L can be given by means of the number of zeros and the number of sign changes of the elements of L. If there exist $t_1, t_2,..., t_{r+1} \in M$ such that $\operatorname{sgn} f(t_i) = -\operatorname{sgn} f(t_{i+1}) \neq 0$ for $i = 1, 2,..., r$, we say that f has r sign changes on M. If f has r but not $r+1$ sign changes, we say that f has exactly r sign changes on M and is denoted

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by $S^-(f, M)$. By $Z(f, M)$ we denote the number of the distinct zeros of f in M.

DEFINITION 2 ([5], Lemma 1]). Let L be a k-dimensional space of real-valued functions defined on M. Then L is a T-space if for every $f \in L\backslash\{0\}$, $Z(f, M) \leqslant k-1$ and $S^-(f, m) \leqslant k-1$.

2. 1-CODIMENSIONAL T-SUBSPACES

We consider T-spaces on a real set M containing its infimum and supremum. Since every 2-dimensional T-space on M contains a positive function, one concludes that it contains a 1-codimensional T-subspace. This is not true in general for T-spaces of higher dimension. In Section 3 we show that for every $n \geqslant 3$ there exists an n-dimensional T-space on a closed interval that does not contain $(n-1)$-dimensional T-subspaces.

We first give a necessary and sufficient condition for a T-space on M to contain a 1-codimensional T-subspace. Let $f_1, f_2, ..., f_{n+2}$, $n \geqslant 1$, be defined on the set M, forming a T-system on it, and let L be their span. If $a = \inf M$ and $b = \sup M$, then

$$L_a = \{f \mid f \in L, f(a) = 0\}, \tag{2}$$

$$L_b = \{f \mid f \in L, f(b) = 0\}, \tag{3}$$

and

$$L_{a,b} = L_a \cap L_b \tag{4}$$

are T-spaces on $M_a = M\backslash\{a\}$, $M_b = M\backslash\{b\}$, and $M_{a,b} = M_a \cap M_b$, respectively.

If L contains a 1-codimensional T-subspace, L', on M, then

$$L'_a = \{f \mid f \in L', f(a) = 0\}, \tag{2'}$$

and

$$L'_b = \{f \mid f \in L', f(b) = 0\} \tag{3'}$$

will be 1-codimensional T-subspaces on M_a and M_b, respectively. If $n \geqslant 2$, then

$$L'_{a,b} = L'_a \cap L'_b \tag{4'}$$

will be a 1-codimensional T-subspace of $L_{a,b}$ on $M_{a,b}$.

We now show that the converse is also true.

THEOREM 1. *Let M be a real set containing both its infimum and supremum. Let $\{f_i\}_{i=1}^{n+2}$ be a set of $n+2$ real-valued functions defined on M and forming a T-system on it and let L_a, L_b, and $L_{a,b}$ be defined as above. Then L contains a* 1*-codimensional T-subspace on M iff*

(i) *L_a and L_b have* 1*-codimensional T-subspaces L'_a and L'_b on M_a and M_b, respectively, and*

(ii) *if $n \geqslant 2$, then $L'_{a,b} = L'_a \cap L'_b$ has dimension $n-1$.*

Remark. If $L'_{a,b}$ is an $(n-1)$-dimensional space, then it is a T-space on $M_{a,b}$.

Proof. We need only show the "if part." Let $\{g_1, g_2,..., g_n, u, v\}$ be a basis of L such that

$$g_i(a) = g_i(b) = 0, \qquad i = 1, 2,..., n, \tag{5}$$

$$u(a) = (-1)^n, \qquad u(b) = 0, \tag{6}$$

and

$$v(a) = 0, \qquad v(b) = 1. \tag{7}$$

Clearly, $L_a = \operatorname{span}\{g_1, g_2,..., g_n, v\}$, $L_b = \operatorname{span}\{g_1, g_2,..., g_n, u\}$ and $L_{a,b} = \operatorname{span}\{g_1, g_2,..., g_n\}$.

Assume first that $n \geqslant 2$. By our hypothesis, we may assume (replacing u and v by $u - \sum_{i=1}^n a_i g_i$ and $v - \sum_{i=1}^n b_i g_i$ if necessary) that

$$L'_a = \operatorname{span}\{g_1, g_2,..., g_{n-1}, v\}, \tag{8}$$

and

$$L'_b = \operatorname{span}\{g_1, g_2,..., g_{n-1}, u\}. \tag{9}$$

Clearly

$$L'_{a,b} = L'_a \cap L'_b. \tag{10}$$

We may also assume that

$$U\begin{pmatrix} g_1, g_2,..., g_n \\ t_1, t_2,..., t_n \end{pmatrix} > 0, \quad \text{whenever} \quad a < t_1 < t_2 < \cdots < t_n < b, \tag{11}$$

and

$$U\begin{pmatrix} g_1, g_2,..., g_{n-1} \\ t_1, t_2,..., t_{n-1} \end{pmatrix} > 0, \quad \text{whenever} \quad a < t_1 < t_2 < \cdots < t_{n-1} < b. \tag{12}$$

Hence

$$U\begin{pmatrix} g_1, g_2,\dots, g_{n-1}, v \\ t_1, t_2,\dots, t_{n-1}, t_n \end{pmatrix} > 0, \quad \text{whenever} \quad a < t_1 < t_2 < \cdots < t_n \leqslant b, \tag{13}$$

$$U\begin{pmatrix} g_1, g_2,\dots, g_n, v \\ t_1, t_2,\dots, t_n, t_{n+1} \end{pmatrix} > 0, \quad \text{whenever} \quad a < t_1 < t_2 < \cdots < t_{n+1} \leqslant b, \tag{14}$$

$$U\begin{pmatrix} g_1, g_2,\dots, g_{n-1}, u \\ t_1, t_2,\dots, t_{n-1}, t_n \end{pmatrix} < 0, \quad \text{whenever} \quad a \leqslant t_1 < t_2 < \cdots < t_n < b, \tag{15}$$

also

$$U\begin{pmatrix} g_1, g_2,\dots, g_n, u \\ t_1, t_2,\dots, t_n, t_{n+1} \end{pmatrix} > 0, \quad \text{whenever} \quad a \leqslant t_1 < t_2 < \cdots < t_{n+1} < b. \tag{16}$$

We now show that $\{g_1, g_2,\dots, g_{n-1}, u, v\}$ forms a T-system on M. Consider the function $h = Au + Bv + \sum_{i=1}^{n-1} c_i g_i$. If $AB = 0$, then $Z(h, M) \leqslant n$ and $S^-(h, M) \leqslant n$. If $AB < 0$, then it follows from (13) and (15) that $Z(h, M_{a,b}) \leqslant n-1$ and $Z(h, M) \leqslant n-1$. Also $S^-(h, M_{a,b}) \leqslant n-1$ and since $h(a) = (-1)^n A$ and $h(b) = B$, one concludes that $S^-(h, M) \leqslant n-1$. Similarly, in case $AB > 0$, (14) and (16) imply that $Z(h, M) \leqslant n$ and $S^-(h, M) \leqslant n$. Hence $\{g_1, g_2,\dots, g_{n-1}, u, v\}$ is a T-system on M. This completes the proof of the theorem for $n \geqslant 2$.

For $n = 1$, L is a 3-dimensional T-space. $L_{a,b}$ is a 1-dimensional space, i.e., $L_{a,b} = \operatorname{span}\{g\}$, where $g(a) = g(b) = 0$. Also, $L_a = \operatorname{span}\{g, v\}$, $L_b = \operatorname{span}\{g, u\}$, $L'_a = \operatorname{span}\{v\}$, and $L'_b = \operatorname{span}\{u\}$ are T-spaces on M_a and M_b, and as before $L' = \operatorname{span}\{u, v\}$ is a T-space on M, which completes the proof of the theorem.

Theorem 1 requires that M contain at least $n+2$ points including its endpoints. We now assume that M has a betweeness property, namely, if $x, y \in M$ with $x < y$, then there exists a point $z \in M$ with $x < z < y$.

Let now $w_a(t) = \max\{\max_{1 \leqslant i \leqslant n} |g_i(t)|, |v(t)|\}$ for $t \in M_a$ and $w_b(t) = \max\{\max_{1 \leqslant i \leqslant n} |g_i(t)|, |u(t)|\}$ for $t \in M_b$. We define

$$y_i(t) = g_i(t)/w_b(t), \qquad i = 1, 2,\dots, n, \quad t \in M_b, \tag{17}$$

$$y_{n+1}(t) = u(t)/w_b(t), \qquad t \in M_b \tag{18}$$

$$z_i(t) = g_i(t)/w_a(t), \qquad i = 1, 2,\dots, n, \quad t \in M_a, \tag{19}$$

and

$$z_{n+1}(t) = v(t)/w_a(t), \qquad t \in M_a. \tag{20}$$

Extend (17)–(20) to M by

$$y_i(b) = \lim_{t \to b'} y_i(t), \qquad i = 1, 2, \ldots, n+1, \tag{21}$$

$$z_i(a) = \lim_{t \to a'} z_i(t), \qquad i = 1, 2, \ldots, n+1, \tag{22}$$

where $a' = \inf M_a$ and $b' = \sup M_b$ (see [2, 3]).

By [2], a necessary and sufficient condition for L_a and L_b to contain a 1-codimensional T-subspace on M_a and M_b, respectively, is that

$$(y_1(a), \ldots, y_{n+1}(a)) \qquad \text{and} \qquad (y_1(b), \ldots, y_{n+1}(b)) \tag{23}$$

are not proportional and also

$$(z_1(a), \ldots, z_{n+1}(a)) \qquad \text{and} \qquad (z_1(b), \ldots, z_{n+1}(b)) \tag{24}$$

are not proportional.

Following the technique of [2, Theorem 1], one finds that $L'_a \cap L'_b$ is an $(n-1)$-dimensional space iff

$$(y_1(b), \ldots, y_n(b)) = \pm(z_1(a), \ldots, z_n(a)) \neq (0, \ldots, 0). \tag{25}$$

We have proved

THEOREM 2. *Let $g_1, g_2, \ldots g_n, u, v$ be as in Theorem* 1. *Let M have the betweeness property and let $y_1, y_2, \ldots, y_{n+1}, z_1, z_2, \ldots, z_{n+1}$ be defined by* (17)–(22). *Then* $\operatorname{span}\{g_1, g_2, \ldots, g_n, u, v\}$ *contains a* 1*-codimensional T-subspace on M iff* (23)–(25) *hold.*

Notice that in [2], continuity had been assumed but it can be easily seen that the results of [2] hold without any continuity assumptions.

3. T-SPACES THAT HAVE NO 1-CODIMENSIONAL T-SUBSPACES

In [4], Zielke shows that for every $n > 2$, there exist n-dimensional T-spaces, on closed and on half open intervals that have no Markov basis. We now show, using the same and analogous examples, that for every $n > 2$ there exists an n-dimensional T-space on a closed interval that has no 1-codimensional T-subspace.

Case 1 (n odd). Consider first the $(n-1)$-dimensional space spanned by

$$f_1(t) = t$$
$$f_i(t) = t^{i-2}(t^2 - 1), \qquad i = 2, 3,..., n-1,$$

defined on $[-1, 1)$.

This is a T-space [4] which does not contain a 1-codimensional T-subspace [2]. Define $p_0 = 1$,

$$p_i(t) = (1-t) f_i(t), \qquad i = 1, 2,..., n-1, \text{ on } [-1, 1].$$

$L = \text{span}\{p_0, p_1,..., p_{n-1}\}$ is a T-space on $[-1, 1]$ (see [4]), and if L contains a 1-codimensional T-subspace, then L_1' (in the notation of Theorem 1) would contain a 1-codimensional T-subspace on $[1, 1)$, which is impossible since L_1' is generated by the f_i's multiplied by a positive function.

Case 2 (n even ($n > 2$)). Similarly, one can show that $L = \text{span}\{p_0, p_1,..., p_{n-1}\}$, where $p_0 = 1$,

$$p_1(t) = 1 - t,$$

and

$$p_i(t) = t^{i-2}(1-t)^2(1+t), \qquad i = 2, 3,..., n-1,$$

is a T-space on $[-1, 1]$ and since $L_1 = \{p \mid p \in L, p(1) = 0\}$ is a T-space (on $[-1, 1)$) that does not contain a 1-codimensional T-subspace [2, Theorem 1], L does not contain a 1-codimensional T-subspace on $[-1, 1]$.

Zielke shows [6, p. 45], that $\text{span}\{f_1,..., f_n, g_1,..., g_n\}$, where $f_i(t) = \sin(it)$ and $g_i(t) = \cos(it)$, $i = 1, 2,..., n$, is a T-space on $[0, \pi]$, for $n \geqslant 2$. He proves that this T-space has no Markov basis and raises the question of whether it contains a 1-codimensional T-subspace or not. Applying Theorem 2, one concludes that it contains no such T-subspace.

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